

# Efficient Clifford+ $T$ approximation of single-qubit operators

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## Abstract

We give an efficient randomized algorithm for approximating an arbitrary element of  $SU(2)$  by a product of Clifford+ $T$  operators, up to any given error threshold  $\epsilon > 0$ . Under a mild hypothesis on the distribution of primes, the algorithm's expected runtime is polynomial in  $\log(1/\epsilon)$ . If the operator to be approximated is a  $z$ -rotation, the resulting gate sequence has  $T$ -count  $K + 4\log_2(1/\epsilon)$ , where  $K$  is approximately equal to 11. We also prove a worst-case lower bound of  $K' + 4\log_2(1/\epsilon)$ , where  $K' = -9$ , so that our algorithm is within an additive constant of optimal for  $z$ -rotations. For an arbitrary member of  $SU(2)$ , we achieve approximations with  $T$ -count  $3K + 12\log_2(1/\epsilon)$ . By contrast, the Solovay-Kitaev algorithm achieves  $T$ -count  $O(\log^c(1/\epsilon))$ , where  $c$  is approximately 3.97.

## 1 Introduction

The decomposition of arbitrary unitary operators into gates from some fixed universal set is a well-known problem in quantum information theory. If the universal gate set is discrete, the decomposition of a general operator can only be done approximately, up to a given accuracy  $\epsilon > 0$ . Here, we focus on the problem of approximating single-qubit operators using the Clifford+ $T$  universal gate set. The Clifford+ $T$  gate set is of particular interest because it is known to be suitable for fault-tolerant quantum computation. Recall that the Clifford group on one qubit is generated by the Hadamard gate  $H$ , the phase gate  $S$ , and the scalar  $\omega = e^{i\pi/4}$ . It is well-known that one obtains a universal gate set by adding the non-Clifford operator  $T$ .

$$\omega = e^{i\pi/4}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}. \quad (1)$$

We present an efficient randomized algorithm for approximating an arbitrary element of  $SU(2)$  by a product of Clifford+ $T$  operators, up to any given error threshold  $\epsilon > 0$ . Under a mild hypothesis on the distribution of primes, the algorithm's expected runtime is polynomial in  $\log(1/\epsilon)$ . The algorithm approximates any  $z$ -rotation with  $T$ -count<sup>1</sup>  $4\log_2(1/\epsilon) + \text{constant}$ , where the additive constant is approximately equal to 11. We also prove a worst-case lower bound of  $K' + 4\log_2(1/\epsilon)$ , where  $K' = -9$ , so that our algorithm is within an additive constant of optimal for  $z$ -rotations. For an arbitrary member of  $SU(2)$ , we achieve approximations with  $T$ -count  $3K + 12\log_2(1/\epsilon)$ .

By contrast, the Solovay-Kitaev algorithm [5, 6, 2], which is the de facto standard algorithm for this problem, produces circuits of  $T$ -count  $O(\log^c(1/\epsilon))$ , where  $c$  is approximately 3.97. Thus, we have decreased the exponent in the gate complexity from  $c = 3.97$  to  $c = 1$ , which is optimal. Moreover, we have decreased the multiplicative constant to the theoretical optimum, in the case of arbitrary  $z$ -rotations.

### 1.1 Related work

For the last 17 years and until a few months ago, the state-of-the-art solution to the approximation problem was the Solovay-Kitaev algorithm [5, 6, 2]. There are various variants of this algorithm. Perhaps the best-known variant is the one described in [2], which achieves a gate complexity of  $O(\log^c(1/\epsilon))$ , for  $c \approx 3.97$ . Another well-known variant is described in Kitaev et al. [6, Sec. 8.3], and achieves  $c = 3 + \delta$ , where  $\delta > 0$  is any positive real number. Kitaev et al. also gave another algorithm that uses ancillas and achieves gate complexity  $O(\log^2(1/\epsilon) \log \log(1/\epsilon))$  [6, Sec. 13.7].

<sup>1</sup>We follow [1] in using  $T$ -count, rather than the overall gate count, as a convenient measure for the length of a single-qubit Clifford+ $T$  circuit. This is justified, on the one hand, because the fault-tolerant implementation of  $T$ -gates is far more resource intensive than that of Clifford gates, and on the other hand, because consecutive Clifford gates can always be combined into a single Clifford gate, so that the overall gate count is almost exactly equal to twice the  $T$ -count.

At the other end of the spectrum, there is a known information-theoretic lower bound of  $c = 1$  for the exponent in the gate complexity. One can make this lower bound more precise: in fact, the decomposition of a typical  $SU(2)$  operator into the Clifford+ $T$  gate set with accuracy  $\epsilon$  requires  $T$ -count at least  $K + 3 \log_2(1/\epsilon)$ , for some constant  $K$ . This follows from a result of Matsumoto and Amano [9], who showed that there are precisely  $192 \cdot (3 \cdot 2^n - 2)$  distinct single-qubit Clifford+ $T$ -circuits of  $T$ -count at most  $n$ , along with the fact that  $SU(2)$  is a 3-dimensional manifold, thus requiring  $\Omega(1/\epsilon^3)$  epsilon-balls to cover. Heuristically, it appears that, for most operators, this lower bound can in fact be achieved by approximation algorithms based on exhaustive search, such as Fowler's algorithm [4]. However, such exhaustive search based algorithms have runtimes that are exponential in  $1/\epsilon$ .

Recently, Duclos-Cianci and Svore announced an alternative to the Solovay-Kitaev algorithm that requires ancillas to be prepared in special resource states, using a state distillation procedure [3]. Using this method, and dependent on the particular setting, they reduced the gate complexity exponent  $c$  to between 1.12 and 2.27.

Even more recently, in an important milestone, Kliuchnikov, Maslov, and Mosca gave an approximation algorithm for single-qubit operators that has polynomial running time and achieves gate counts of  $O(\log(1/\epsilon))$ , thus reducing the gate complexity exponent to  $c = 1$  [7]. The Kliuchnikov-Maslov-Mosca approximation algorithm is therefore asymptotically optimal. It uses a fixed, small number of ancilla qubits to approximate a given single-qubit operator. Unlike approaches based on resource states, the ancillas in the Kliuchnikov-Maslov-Mosca algorithm are initialized to  $|0\rangle$ , and are returned in a state very close (but not exactly equal) to  $|0\rangle$ ; these ancillas do not require any special preparation procedure.

For all practical purposes, this use of ancillas in the Kliuchnikov-Maslov-Mosca algorithm causes no difficulties. However, the question remained open whether there exists an asymptotically optimal, efficient single-qubit approximation algorithm that requires no ancillas, and thus solves exactly the same problem for the Clifford+ $T$  gate set as the Solovay-Kitaev algorithm. The present paper achieves such an algorithm.

## 1.2 Limitations

Unlike the Solovay-Kitaev algorithm, which can be applied to any universal gate set, the algorithm of this paper is specialized to the Clifford+ $T$  gate set. It is an open question whether a similar approach would work for other universal gate sets as well.

Technically, the expected polynomial runtime of the present algorithm is contingent on a number-theoretic assumption about the distribution of primes, as stated below in Hypothesis 28. I am not an expert on the distribution of prime numbers, and do not know, as of this writing, whether this hypothesis is known to be true or not. Therefore, this has been left open.

The lower bound of  $K' + 4 \log_2(1/\epsilon)$  for the  $T$ -count of  $\epsilon$ -approximations to certain  $z$ -rotations only applies to the problem as stated, i.e., for writing operators as a product of  $2 \times 2$ -Clifford+ $T$  operators. It is conceivable that by the use of other techniques, such as ancillas, resource states, or online measurement, even smaller  $T$ -counts can be obtained. However, no method for doing so is currently known.

## 2 Overview of the algorithm

The algorithm can be summarized as follows. Consider the ring

$$\mathbb{D}[\omega] = \mathbb{Z}[\frac{1}{\sqrt{2}}, i] = \{ \frac{1}{\sqrt{2}^k} (a\omega^3 + b\omega^2 + c\omega + d) \mid k \in \mathbb{N}; a, b, c, d \in \mathbb{Z} \}. \quad (2)$$

As shown by Kliuchnikov, Maslov, and Mosca [8], a unitary operator  $U \in U(2)$  can be exactly synthesized over the Clifford+ $T$  gate set (with no ancillas) if and only if all the matrix entries belong to the ring  $\mathbb{D}[\omega]$ . Moreover, the required number of  $T$ -gates is at most  $2k$ , where  $k \geq 0$  is the minimal exponent required to write all entries of  $U$  in the form mentioned in (2).

Suppose we wish to approximate a given  $z$ -rotation

$$R_z(\theta) = e^{-i\theta Z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}, \quad (3)$$

up to a given  $\epsilon > 0$ , using Clifford+ $T$  gates. We will choose an integer  $k$  and a randomized sequence of suitable elements  $u \in \mathbb{Z}[\omega]$  such that  $\frac{u}{\sqrt{2}^k} \approx e^{-i\theta/2}$ . For each  $u$ , we attempt to solve the Diophantine equation

$$tt^\dagger + uu^\dagger = 2^k, \quad (4)$$

with  $t \in \mathbb{Z}[\omega]$ . The parameters are chosen in such a way that this succeeds for a relatively large proportion of the available  $u$ . This yields a unitary matrix

$$U = \frac{1}{\sqrt{2}^k} \begin{pmatrix} u & -t^\dagger \\ t & u^\dagger \end{pmatrix} \quad (5)$$

with  $\|U - R_z(\theta)\| \leq \epsilon$ , and such that the coefficients of  $U$  are in the ring  $\mathbb{D}[\omega]$ . By the Kliuchnikov-Maslov-Mosca exact synthesis algorithm,  $U$  can be exactly decomposed into Clifford+ $T$  gates with  $T$ -count at most  $2k$ . The remainder of this paper fills in the details of these ideas: in particular, how to choose  $k$ , how to find “suitable”  $u$ , and how to solve the Diophantine equation (4) with high probability.

### 3 Some algebra

#### 3.1 Some rings of algebraic integers

Recall that  $\mathbb{N}$  is the set of natural numbers including 0. Let  $\omega = e^{i\pi/4} = (1+i)/\sqrt{2}$ . Note that  $\omega$  is an 8th root of unity satisfying  $\omega^2 = i$  and  $\omega^4 = -1$ . We will consider the following rings:

- $\mathbb{Z}$ , the ring of integers;
- $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ , the ring of *quadratic integers with radicand 2*;
- $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ , the ring of *Gaussian integers*;
- $\mathbb{Z}[\omega] = \{a\omega^3 + b\omega^2 + c\omega + d \mid a, b, c, d \in \mathbb{Z}\}$ , the ring of *cyclotomic integers of degree 8*.

**Remark 1.** We have the inclusions  $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{2}] \subseteq \mathbb{Z}[\omega]$  and  $\mathbb{Z} \subseteq \mathbb{Z}[i] \subseteq \mathbb{Z}[\omega]$ . Of course, all four rings are subrings of the complex numbers.

#### 3.2 Conjugate and norm

**Definition 2** (Conjugation). Since  $\omega$  is a 4th root of  $-1$ , the ring  $\mathbb{Z}[\omega]$  has four automorphisms. One of these is the usual *complex conjugation*, which we denote  $(-)^{\dagger}$ . It maps  $i$  to  $-i$ , and  $\sqrt{2}$  to itself. Equivalently, it is given by  $\omega^{\dagger} = -\omega^3$ . It acts trivially in  $\mathbb{Z}$  and  $\mathbb{Z}[\sqrt{2}]$ , and non-trivially on  $\mathbb{Z}[\omega]$  and  $\mathbb{Z}[i]$ , with the following explicit formulas, for real  $a, b, c, d$ :

$$(a\omega^3 + b\omega^2 + c\omega + d)^{\dagger} = -c\omega^3 - b\omega^2 - a\omega + d, \quad (6)$$

$$(a + bi)^{\dagger} = a - bi. \quad (7)$$

Another automorphism is  $\sqrt{2}$ -conjugation, which we denote  $(-)^{\bullet}$ . It maps  $\sqrt{2}$  to  $-\sqrt{2}$ , and  $i$  to itself. Equivalently,  $\omega^{\bullet} = -\omega$ . It acts trivially on  $\mathbb{Z}$  and  $\mathbb{Z}[i]$ , and non-trivially on  $\mathbb{Z}[\omega]$  and  $\mathbb{Z}[\sqrt{2}]$ , with the following explicit formulas, for rational  $a, b, c, d$ :

$$(a\omega^3 + b\omega^2 + c\omega + d)^{\bullet} = -a\omega^3 + b\omega^2 - c\omega + d, \quad (8)$$

$$(a + b\sqrt{2})^{\bullet} = a - b\sqrt{2}. \quad (9)$$

The remaining two automorphisms are, of course, the identity function and  $(-)^{\dagger\bullet} = (-)^{\bullet\dagger}$ .

**Remark 3.** For  $t \in \mathbb{Z}[\omega]$ , we have  $t \in \mathbb{Z}[\sqrt{2}]$  iff  $t = t^{\dagger}$ ,  $t \in \mathbb{Z}[i]$  iff  $t = t^{\bullet}$ , and  $t \in \mathbb{Z}$  iff  $t = t^{\dagger}$  and  $t = t^{\bullet}$ .

**Definition 4** (Norms). We define an integer-valued (number-theoretic) *norm* on each ring:

- For  $t = a + bi \in \mathbb{Z}[i]$ , let

$$\mathcal{N}_i(t) = tt^{\dagger} = a^2 + b^2. \quad (10)$$

- For  $t = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ , let

$$\mathcal{N}_{\sqrt{2}}(t) = tt^{\bullet} = a^2 - 2b^2. \quad (11)$$

- For  $t = a\omega^3 + b\omega^2 + c\omega + d \in \mathbb{Z}[\omega]$ , let

$$\mathcal{N}_{\omega}(t) = tt^{\dagger}(tt^{\dagger})^{\bullet} = (a^2 + b^2 + c^2 + d^2)^2 - 2(ab + bc + cd - da)^2. \quad (12)$$

For consistency, we also define  $\mathcal{N}_1(t) = t$  for  $t \in \mathbb{Z}$ .

**Remark 5.**  $\mathcal{N}_i$  and  $\mathcal{N}_\omega$  are valued in the non-negative integers, but  $\mathcal{N}_1$  and  $\mathcal{N}_{\sqrt{2}}$  may take negative values. Each norm is multiplicative, in the sense that  $\mathcal{N}(ts) = \mathcal{N}(t)\mathcal{N}(s)$  for all  $t, s$ . Moreover,  $\mathcal{N}(t) = 0$  if and only if  $t = 0$ , and  $\mathcal{N}(t) = \pm 1$  if and only if  $t$  is a unit (i.e., an invertible element) in the ring. The latter property follows easily from multiplicativity and the fact that equations (10–12) define an inverse for  $t$  when  $\mathcal{N}(t) = \pm 1$ .

### 3.3 Euclidean domains

**Remark 6.**  $\mathbb{Z}$ ,  $\mathbb{Z}[\sqrt{2}]$ ,  $\mathbb{Z}[i]$ , and  $\mathbb{Z}[\omega]$  are Euclidean domains. Explicitly, for each of these rings, a Euclidean function is given by  $|\mathcal{N}(t)|$ . For given elements  $s$  and  $t \neq 0$ , the division of  $s$  by  $t$  with quotient  $q$  and remainder  $r$  can be defined by first calculating  $q' = s/t$  in the corresponding field of fractions (respectively  $\mathbb{Q}$ ,  $\mathbb{Q}[\sqrt{2}]$ ,  $\mathbb{Q}[i]$ , and  $\mathbb{Q}[\omega]$ ).  $q'$  can be expressed with rational coefficients as  $a$ ,  $a + b\sqrt{2}$ ,  $a + bi$ , or  $\omega^3a + \omega^2b + \omega c + d$ , respectively. Then  $q$  is obtained from  $q'$  by rounding each coefficient  $a, b, c, d$  to the closest integer, and  $r$  is defined to be  $qt - s$ . One may verify that in each case,  $|\mathcal{N}(r)| \leq \frac{9}{16}|\mathcal{N}(t)|$ .

As usual, we write  $t | s$  to mean that  $t$  is a divisor of  $s$ , i.e., that there exists some  $r$  such that  $rt = s$ . We also write  $t \sim s$  to indicate that  $t | s$  and  $s | t$ ; in this case,  $t$  and  $s$  differ only by a unit of the ring, and we say  $t$  and  $s$  are *associates*.

**Remark 7.** We note that if  $R$  is one of the four rings  $\mathbb{Z}$ ,  $\mathbb{Z}[\sqrt{2}]$ ,  $\mathbb{Z}[i]$ , and  $\mathbb{Z}[\omega]$ , and  $rt = s$  for some  $s, t \in R$  and  $r \in \mathbb{Z}[\omega]$  with  $t \neq 0$ , then  $r \in R$ . This is easily proved by letting  $\bar{R}$  be the corresponding field of fractions (respectively  $\mathbb{Q}$ ,  $\mathbb{Q}[\sqrt{2}]$ ,  $\mathbb{Q}[i]$ , and  $\mathbb{Q}[\omega]$ ), and noting on the one hand that  $r = \frac{s}{t} \in \bar{R}$ , and on the other hand that  $R = \mathbb{Z}[\omega] \cap \bar{R}$ . The latter property follows from Remark 3.

In particular, it follows that  $t | s$  holds in  $R$  if and only if  $t | s$  holds in  $\mathbb{Z}[\omega]$ , so that the notion of divisibility is independent of  $R$ .

**Remark 8.** Every Euclidean domain admits greatest common divisors, which can be calculated by repeated divisions with remainder via Euclid's algorithm. Note that greatest common divisors are only unique up to a unit of the ring. Also note that, since each division by  $t$  with remainder  $r$  satisfies  $|\mathcal{N}(r)| \leq \frac{9}{16}|\mathcal{N}(t)|$ , the computation of the greatest common divisor of two elements of any of the above rings requires at most  $O(\log |\mathcal{N}(t)|)$  divisions with remainder.

An element  $t$  of a Euclidean domain is called *prime* (or *irreducible*) if  $t$  is not a unit, and for all  $r, s$  with  $rs = t$ , either  $r$  or  $s$  is a unit. We note that the notion of primality is not independent of the ring. For example, 7 is prime in  $\mathbb{Z}$ , but not in  $\mathbb{Z}[\sqrt{2}]$ , as  $7 = (3 + \sqrt{2})(3 - \sqrt{2})$ .

**Remark 9.** In each of the above rings, if  $\mathcal{N}(t)$  is prime in  $\mathbb{Z}$ , then  $t$  is prime in the ring. Indeed, suppose  $t = rs$ . Since  $\mathcal{N}(t) = \mathcal{N}(r)\mathcal{N}(s)$ , either  $\mathcal{N}(r)$  or  $\mathcal{N}(s)$  is  $\pm 1$ , hence  $r$  or  $s$  is a unit.

### 3.4 Units in $\mathbb{Z}[\sqrt{2}]$

**Lemma 10.** *The units of  $\mathbb{Z}[\sqrt{2}]$  are exactly the elements of the form  $(-1)^n(\sqrt{2} - 1)^k$ , where  $n, k \in \mathbb{Z}$ . A unit  $u$  is a square if and only if  $u \geq 0$  and  $u^\bullet \geq 0$ .*

*Proof.* We first note that  $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$ , so  $\sqrt{2} - 1$  and  $\sqrt{2} + 1$  are units. Now consider any unit  $u = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ . We prove the first claim by induction on  $|b|$ . The base case is  $b = 0$ ; in this case,  $\mathcal{N}_{\sqrt{2}}(u) = a^2 = \pm 1$  implies  $u = a = \pm 1$ . For the induction step, note that  $\mathcal{N}_{\sqrt{2}}(u) = a^2 - 2b^2 = \pm 1$  implies  $a \neq 0$  and  $a^2 < 2b^2 + 2 \leq 4b^2$ , hence  $0 < |a| < |2b|$ . First consider the case where  $a, b$  have the same sign. Then  $|a - b| < |b|$ , and the claim is proved by applying the induction hypothesis to  $u' = u(\sqrt{2} - 1) = 2b - a + (a - b)\sqrt{2}$ . The case where  $a, b$  have opposite signs is similar, except we use  $|a + b| < |b|$  to apply the induction hypothesis to  $u' = u/(\sqrt{2} - 1) = a + 2b + (a + b)\sqrt{2}$ .

For the second claim, note that  $u = (-1)^n(\sqrt{2} - 1)^k$  satisfies  $u \geq 0$  iff  $n$  is even, and  $u^\bullet \geq 0$  iff  $n + k$  is even. Therefore  $u$  is a square iff  $n$  and  $k$  are both even, iff  $u \geq 0$  and  $u^\bullet \geq 0$ .  $\square$

### 3.5 Roots of $-1$ in $\mathbb{Z}/(p)$

**Remark 11.** Let  $p \in \mathbb{Z}$  be a positive prime satisfying  $p \equiv 1 \pmod{4}$ . It is well-known that there exists  $h \in \mathbb{Z}$  such that  $h^2 + 1 = 0 \pmod{p}$ . We recall that there is an efficient randomized algorithm for computing  $h$ . Consider the field  $\mathbb{Z}/(p)$  of integers modulo  $p$ . By Fermat's Little Theorem, all non-zero  $b \in \mathbb{Z}/(p)$  satisfy  $b^{p-1} = 1$ , hence  $b^{(p-1)/2} = \pm 1$ . Because each of the polynomial equations  $b^{(p-1)/2} = 1$  and  $b^{(p-1)/2} = -1$  has at most  $(p-1)/2$  solutions,  $b^{(p-1)/2} = -1$  holds for exactly half of the non-zero  $b \in \mathbb{Z}/(p)$ . Therefore, one can efficiently solve

$b^{(p-1)/2} = -1$  by picking  $b$  at random until a solution is found; on average, this will require two attempts. Note that, by the method of repeated squaring, the computation of  $b^{(p-1)/2}$  only requires  $O(\log p)$  multiplications. Finally, we can set  $h = b^{(p-1)/4}$ .

## 4 A Diophantine equation

We will be interested in solving equations of the form

$$tt^\dagger = \xi, \quad (13)$$

where  $\xi \in \mathbb{Z}[\sqrt{2}]$  is given and  $t \in \mathbb{Z}[\omega]$  is unknown. Writing  $\xi = x + y\sqrt{2}$  and  $t = a\omega^3 + b\omega^2 + c\omega + d$ , we can equivalently express (13) as a pair of integer equations:

$$a^2 + b^2 + c^2 + d^2 = x \quad (14)$$

$$ab + bc + cd - da = y. \quad (15)$$

Of course, not every  $\xi \in \mathbb{Z}[\sqrt{2}]$  can be expressed in the form (13). However, we have the following:

**Theorem 12.** *Let  $\xi = x + y\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ , where  $x$  is odd,  $y$  is even,  $\xi \geq 0$ ,  $\xi^\bullet \geq 0$ , and  $p = \xi\xi^\bullet = x^2 - 2y^2$  is prime in  $\mathbb{Z}$ . Then there exists  $t \in \mathbb{Z}[\omega]$  satisfying (13). Moreover, there is an efficient randomized algorithm for computing  $t$ .*

**Remark 13.** Since  $tt^\dagger = \xi$  implies  $t^\bullet t^{\bullet\dagger} = \xi^\bullet$ , the conditions  $\xi \geq 0$  and  $\xi^\bullet \geq 0$  are both obviously necessary for (13) to have a solution.

*Proof of Theorem 12.* Note that  $p$  is prime by assumption. Since  $\xi \geq 0$  and  $\xi^\bullet \geq 0$ , we know that  $p \geq 0$ . Since  $x$  is odd and  $y$  is even, we have  $p \equiv 1 \pmod{4}$ . Therefore, by Remark 11, we can find  $h \in \mathbb{Z}$  with  $p \mid h^2 + 1$ . Moreover, in  $\mathbb{Z}[\sqrt{2}]$ , we have  $\xi \mid p$  and therefore  $\xi \mid h^2 + 1$ . Let  $s = \gcd(h + i, \xi)$  in the ring  $\mathbb{Z}[\omega]$ . We claim that  $ss^\dagger \sim \xi$ .

First, note that  $\xi$  is prime in the ring  $\mathbb{Z}[\sqrt{2}]$  by Remark 9. By definition of  $s$ , we know that  $s \mid \xi$ . But  $\xi$  is real, and therefore also  $s^\dagger \mid \xi$ . It follows that  $ss^\dagger \mid \xi^2$  in  $\mathbb{Z}[\omega]$ . By Remark 7,  $ss^\dagger \mid \xi^2$  in  $\mathbb{Z}[\sqrt{2}]$ . Since  $\xi$  is prime in  $\mathbb{Z}[\sqrt{2}]$ , it follows that one of three cases holds:

- (a)  $ss^\dagger \sim 1$ . But this is impossible. Indeed, in this case,  $s$  is a unit, so  $h + i$  and  $\xi$  are relatively prime. As  $\xi$  is real, it follows that  $h - i$  and  $\xi$  are also relatively prime, hence  $(h + i)(h - i)$  is relatively prime to  $\xi$ , contradicting  $\xi \mid h^2 + 1$ .
- (b)  $ss^\dagger \sim \xi$ . This is what was claimed.
- (c)  $ss^\dagger \sim \xi^2$ . This is also impossible. Indeed, in this case,  $\mathcal{N}_\omega(s) = \mathcal{N}_{\sqrt{2}}(ss^\dagger) = \mathcal{N}_{\sqrt{2}}(\xi^2) = \mathcal{N}_\omega(\xi)$ . But we also have  $s \mid \xi$ , so  $\xi = us$  for some  $u \in \mathbb{Z}[\sqrt{2}]$ . Then  $\mathcal{N}_\omega(u) = \mathcal{N}_\omega(\xi)/\mathcal{N}_\omega(s) = 1$ , hence  $u$  is a unit in  $\mathbb{Z}[\sqrt{2}]$ , thus  $s \sim \xi$ . Since, by definition,  $s \mid h + i$ , we have  $\xi \mid h + i$ . Since  $h$  is an integer, we have  $h = h^\bullet$ , and thus  $\xi^\bullet \mid h + i$ . We note that  $\xi \not\sim \xi^\bullet$ , for otherwise, we would have  $p = \xi\xi^\bullet \mid \xi^2 + \xi^{\bullet 2} = 2x^2 + 4y^2 = 4x^2 - 2p$ ; since  $p$  is an odd prime, this implies  $p \mid x$ , and from  $p = x^2 - 2y^2$ , we get  $p \mid y$ , hence  $p^2 \mid x^2 - 2y^2 = p$ , a contradiction. Therefore,  $\xi$  and  $\xi^\bullet$  are non-associate primes in  $\mathbb{Z}[\sqrt{2}]$ , both dividing  $h + i$ , which implies  $p = \xi\xi^\bullet \mid h + i$ , an absurdity since  $\frac{1}{p}(h + i) \notin \mathbb{Z}[\omega]$ .

We have shown the existence of  $s \in \mathbb{Z}[\omega]$  such that  $ss^\dagger \sim \xi$  in  $\mathbb{Z}[\sqrt{2}]$ . Let  $u$  be a unit of  $\mathbb{Z}[\sqrt{2}]$  such that  $uss^\dagger = \xi$ . Our next claim is that  $u$  is a square in  $\mathbb{Z}[\sqrt{2}]$ . First note that, by the usual properties of complex numbers,  $ss^\dagger \geq 0$  and  $(ss^\dagger)^\bullet = (s^\bullet)(s^\bullet)^\dagger \geq 0$ . Also,  $\xi \geq 0$  and  $\xi^\bullet \geq 0$  by assumption. It follows that  $u \geq 0$  and  $u^\bullet \geq 0$ . But then  $u$  is a square by Lemma 10. Let  $v \in \mathbb{Z}[\sqrt{2}]$  with  $v^2 = u$ , and let  $t = vs$ . Noting that  $v = v^\dagger$ , we have  $tt^\dagger = v^2ss^\dagger = \xi$ , as desired.  $\square$

**Remark 14.** We note that the proof of Theorem 12 immediately yields an algorithm for computing  $t$ . As a matter of fact, the only randomized aspect is the computation of  $h$ ; the remainder consists of arithmetic calculations in the various rings that can be done deterministically. The computation of  $s$  requires taking a greatest common divisor in  $\mathbb{Z}[\omega]$ , which can be done efficiently by Remark 8. The computation of  $u$  requires a simple division in  $\mathbb{Z}[\sqrt{2}]$ , and the computation of  $v$  requires taking a square root in  $\mathbb{Z}[\sqrt{2}]$ , which easily reduces to solving a quadratic equation in the integers. Finally, the computation of  $t$  requires a multiplication in  $\mathbb{Z}[\omega]$ .

**Remark 15.** Theorem 12 is analogous to a well-known theorem about the integers, stating that for every positive prime satisfying  $p \equiv 1 \pmod{4}$ , the equation  $zz^\dagger = p$  has a solution with  $z \in \mathbb{Z}[i]$ . A randomized algorithm for computing  $z$  was described, for example, by Rabin and Shallit [10]. Our proof and algorithm follows the same general idea, applied to a different pair of Euclidean rings.

## 5 Approximations in $\mathbb{Z}[\sqrt{2}]$

It is well-known that the set  $\mathbb{Z}[\sqrt{2}]$  of integers of the form  $\alpha = a + b\sqrt{2}$  is a dense subset of the real numbers. Here, density is of course understood with respect to the Euclidean distance  $|\alpha - \beta|$ . We note that the Euclidean distance is not at all preserved by  $\sqrt{2}$ -conjugation; in fact, as we will see in Lemma 16 below, unless  $\alpha = \beta$ , it is impossible for  $|\alpha - \beta|$  and  $|\alpha^\bullet - \beta^\bullet|$  to be small at the same time.

The purpose of this section is to find solutions in  $\mathbb{Z}[\sqrt{2}]$  to constraints involving  $\alpha$  and  $\alpha^\bullet$  simultaneously. Specifically, we will be interested in solving problems of the form

$$a + b\sqrt{2} \in [x_0, x_1] \quad \text{and} \quad a - b\sqrt{2} \in [y_0, y_1], \quad (16)$$

where  $x_0 < x_1$  and  $y_0 < y_1$  are given real numbers, and  $a, b$  are unknown integers. We start with a result limiting the number of solutions.

**Lemma 16.** *Let  $[x_0, x_1]$  and  $[y_0, y_1]$  be closed intervals of real numbers. Let  $\delta = x_1 - x_0$  and  $\Delta = y_1 - y_0$ , and assume  $\delta\Delta < 1$ . Then there exists at most one  $\alpha = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$  satisfying (16).*

*Proof.* Suppose  $\alpha$  and  $\beta$  are two solutions. Then

$$|\mathcal{N}_{\sqrt{2}}(\alpha - \beta)| = |\alpha - \beta| \cdot |\alpha^\bullet - \beta^\bullet| \leq \delta\Delta < 1. \quad (17)$$

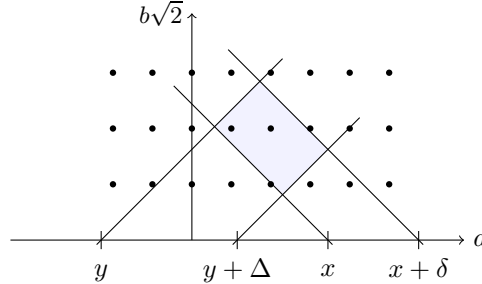
Since  $\mathcal{N}_{\sqrt{2}}(\alpha - \beta)$  is an integer, it follows that  $\mathcal{N}_{\sqrt{2}}(\alpha - \beta) = 0$ , and therefore  $\alpha = \beta$ .  $\square$

The next result establishes the existence of solutions.

**Lemma 17.** *Let  $[x_0, x_1]$  and  $[y_0, y_1]$  be closed intervals of real numbers. Let  $\delta = x_1 - x_0$  and  $\Delta = y_1 - y_0$ , and assume  $\delta\Delta \geq (1 + \sqrt{2})^2$ . Then there exists at least one  $\alpha = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$  satisfying (16). Moreover, there is an efficient algorithm for computing such  $a$  and  $b$ .*

*Proof.* Let us say that a pair of positive real numbers  $(\delta, \Delta)$  has the *coverage property* if for all  $x, y \in \mathbb{R}$ , there exists  $\alpha \in \mathbb{Z}[\sqrt{2}]$  with  $(\alpha, \alpha^\bullet) \in [x, x + \delta] \times [y, y + \Delta]$ . The goal, then, is to show that  $\delta\Delta \geq (1 + \sqrt{2})^2$  implies the coverage property.

Before we continue, it is perhaps helpful to consider the following illustration. Here,  $a$  is shown on the horizontal axis, and  $b\sqrt{2}$  is shown on the vertical axis. The region defined by  $(a + b\sqrt{2}, a - b\sqrt{2}) \in [x, x + \delta] \times [y, y + \Delta]$  is a rectangle oriented at 45 degrees. We are only interested in solutions where  $a, b$  are integers; these solutions therefore lie on the grid  $\mathbb{Z} \times \sqrt{2}\mathbb{Z}$ . Note that the horizontal and vertical spacing are not the same.



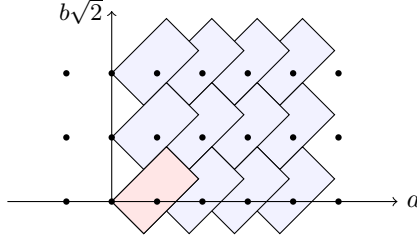
We prove a sequence of claims leading up to the lemma.

- (a) If  $(\delta, \Delta)$  has the coverage property and  $\delta' \geq \delta$ ,  $\Delta' \geq \Delta$ , then  $(\delta', \Delta')$  has the coverage property. This is a trivial consequence of the definitions.
- (b) For all  $\delta, \Delta > 0$ , the pair  $(\delta, \Delta)$  has the coverage property if and only if  $(\Delta, \delta)$  has the coverage property. This is a trivial consequence of the definitions.
- (c) Let  $\lambda = \sqrt{2} - 1$ . Then for all  $\delta, \Delta > 0$ , the pair  $(\delta, \Delta)$  has the coverage property if and only if  $(\lambda\delta, \lambda^{-1}\Delta)$  has the coverage property. Indeed, note that  $0 < \lambda < 1$ , and recall that  $\lambda^{-1} = -\lambda^\bullet = \sqrt{2} + 1$ . Therefore both  $\lambda$  and  $\lambda^{-1}$  are elements of  $\mathbb{Z}[\sqrt{2}]$ . Suppose  $(\delta, \Delta)$  has the coverage property, and  $x, y \in \mathbb{R}$  are given. Then there exists  $\alpha \in \mathbb{Z}[\sqrt{2}]$  with  $(\alpha, \alpha^\bullet) \in [\lambda^{-1}x, \lambda^{-1}x + \delta] \times [-\lambda y - \Delta, -\lambda y]$ . Let  $\beta = \lambda\alpha$ . It follows that

$$(\beta, \beta^\bullet) = (\lambda\alpha, -\lambda^{-1}\alpha^\bullet) \in [x, x + \lambda\delta] \times [y, y + \lambda^{-1}\Delta], \quad (18)$$

so  $(\lambda\delta, \lambda^{-1}\Delta)$  has the coverage property. The converse is proved symmetrically.

- (d)  $(\delta, \Delta) = (1 + \sqrt{2}, \sqrt{2})$  has the coverage property. Geometrically, this is equivalent to the statement that  $\mathbb{R}^2$  is covered by all translations along the grid  $\mathbb{Z} \times \sqrt{2}\mathbb{Z}$  of the rectangle  $R$  defined by  $(a + b\sqrt{2}, a - b\sqrt{2}) \in [0, 1 + \sqrt{2}] \times [0, \sqrt{2}]$ .



In order to have an explicit algorithm for finding  $\alpha$ , we give an algebraic proof. Let  $a, b$  be integers such that

$$a - 1 \leq \frac{x + y + \Delta}{2} < a, \quad (19)$$

$$(b - 1)\sqrt{2} \leq \frac{x - y - \Delta}{2} < b\sqrt{2}. \quad (20)$$

Let  $\alpha = a + b\sqrt{2}$ ,  $\alpha' = a + (b + 1)\sqrt{2}$ , and  $\alpha'' = (a - 1) + b\sqrt{2}$ . We claim that either  $\alpha$ ,  $\alpha'$ , or  $\alpha''$  is a solution to (16).

- Case 1:  $a - b\sqrt{2} \leq y + \Delta$ . In this case, we have:

$$\alpha = a + b\sqrt{2} > \frac{x + y + \Delta}{2} + \frac{x - y - \Delta}{2} = x \quad \text{by (19) and (20),}$$

$$\alpha = a + b\sqrt{2} \leq \frac{x + y + \Delta}{2} + 1 + \frac{x - y - \Delta}{2} + \sqrt{2} = x + \delta \quad \text{by (19) and (20),}$$

$$\alpha^\bullet = a - b\sqrt{2} > \frac{x + y + \Delta}{2} - \frac{x - y - \Delta}{2} - \sqrt{2} = y \quad \text{by (19) and (20),}$$

$$\alpha^\bullet = a - b\sqrt{2} \leq y + \Delta \quad \text{by assumption.}$$

Therefore,  $\alpha$  is a solution to (16).

- Case 2:  $a - b\sqrt{2} > y + \Delta$  and  $a + b\sqrt{2} \leq x + 1$ . In this case, we have:

$$\alpha' = a + (b + 1)\sqrt{2} > \frac{x + y + \Delta}{2} + \frac{x - y - \Delta}{2} + \sqrt{2} > x \quad \text{by (19) and (20),}$$

$$\alpha' = a + (b + 1)\sqrt{2} \leq x + 1 + \sqrt{2} = x + \delta \quad \text{by assumption,}$$

$$\alpha'^\bullet = a - (b + 1)\sqrt{2} > y + \Delta - \sqrt{2} = y \quad \text{by assumption,}$$

$$\alpha'^\bullet = a - (b + 1)\sqrt{2} < \frac{x + y + \Delta}{2} + 1 - \frac{x - y - \Delta}{2} - \sqrt{2} = y + 1 \leq y + \Delta \quad \text{by (19) and (20).}$$

Therefore,  $\alpha'$  is a solution to (16).

- Case 3:  $a - b\sqrt{2} > y + \Delta$  and  $a + b\sqrt{2} > x + 1$ . In this case, we have:

$$\alpha'' = (a - 1) + b\sqrt{2} > x \quad \text{by assumption,}$$

$$\alpha'' = (a - 1) + b\sqrt{2} \leq \frac{x + y + \Delta}{2} + \frac{x - y - \Delta}{2} + \sqrt{2} < x + \delta \quad \text{by (19) and (20),}$$

$$\alpha''^\bullet = (a - 1) - b\sqrt{2} > y + \Delta - 1 > y \quad \text{by assumption,}$$

$$\alpha''^\bullet = (a - 1) - b\sqrt{2} \leq \frac{x + y + \Delta}{2} - \frac{x - y - \Delta}{2} = y + \Delta \quad \text{by (19) and (20).}$$

Therefore,  $\alpha''$  is a solution to (16).

- (e)  $(\delta, \Delta) = (2 + \sqrt{2}, 1)$  has the coverage property. This follows from (b)–(d), by noting that  $2 + \sqrt{2} = \lambda^{-1}\sqrt{2}$  and  $1 = \lambda(1 + \sqrt{2})$ .

- (f) Suppose  $\delta\Delta \geq (1 + \sqrt{2})^2$  and  $1 < \delta \leq 1 + \sqrt{2}$ . Then  $(\delta, \Delta)$  has the coverage property. We consider two cases. Case 1:  $\delta > \sqrt{2}$ . Since  $\delta \leq 1 + \sqrt{2}$ , we have  $\Delta \geq 1 + \sqrt{2}$ . But  $(\sqrt{2}, 1 + \sqrt{2})$  has the coverage property by (d), so the claim follows from (a). Case 2:  $\delta \leq \sqrt{2}$ . Then

$$\Delta \geq \frac{(1 + \sqrt{2})^2}{\sqrt{2}} > 2 + \sqrt{2}. \quad (21)$$

Also  $\delta > 1$ . But  $(1, 2 + \sqrt{2})$  has the coverage property by (e), so the claim follows from (a).

- (g) Suppose  $\delta, \Delta > 0$  such that  $\delta\Delta \geq (1 + \sqrt{2})^2$ . Then  $(\delta, \Delta)$  has the coverage property. Indeed, note that there exists some  $n \in \mathbb{Z}$  such that  $1 < \lambda^n \delta \leq \lambda^{-1} = 1 + \sqrt{2}$ . Then  $(\lambda^n \delta, \lambda^{-n} \Delta)$  has the coverage property by (f), and  $(\delta, \Delta)$  has the coverage property by (c).

This finishes the proof of the lemma. Note that in each step of the proof, we have shown explicitly how to compute a solution  $\alpha$ . Since there are no iterative constructions (apart from the computation of  $\lambda^n$  in (g), which can be done in  $O(|\log(\delta)|)$  steps), the computational effort is essentially trivial, and certainly not greater than say  $O(\log^2(\delta))$ .  $\square$

**Remark 18.** We note that the bounds on  $\delta\Delta$  in Lemmas 16 and 17 are sharp. For the sharpness of Lemma 16, consider  $(\alpha, \alpha^\bullet) \in [0, 1] \times [0, 1]$ , which has two solutions  $\alpha \in \{0, 1\}$ . For the sharpness of Lemma 17, consider  $(\alpha, \alpha^\bullet) \in [0, 1 + \sqrt{2}] \times [-\sqrt{2}, 1]$ , which has exactly four solutions  $\alpha \in \{0, 1, \sqrt{2}, 1 + \sqrt{2}\}$ . Therefore, for all  $\epsilon > 0$ ,  $(\alpha, \alpha^\bullet) \in [\epsilon, 1 + \sqrt{-\epsilon}] \times [-\sqrt{2} + \epsilon, 1 - \epsilon]$  has no solutions at all.

We also give a version of Lemma 17 where  $a$  is restricted to be either even or odd:

**Corollary 19.** Let  $[x_0, x_1]$  and  $[y_0, y_1]$  be closed intervals of real numbers. Let  $\delta = x_1 - x_0$  and  $\Delta = y_1 - y_0$ , and assume  $\delta\Delta \geq 2(1 + \sqrt{2})^2$ . Then there exist  $a', b' \in \mathbb{Z}$  such that  $a' + b'\sqrt{2} \in [x_0, x_1]$  and  $a' - b'\sqrt{2} \in [y_0, y_1]$ , and  $a'$  is even. There also exist  $a'', b'' \in \mathbb{Z}$  such that  $a'' + b''\sqrt{2} \in [x_0, x_1]$  and  $a'' - b''\sqrt{2} \in [y_0, y_1]$ , and  $a''$  is odd. Moreover, there is an efficient algorithm for computing such  $a', b', a'',$  and  $b''$ .

*Proof.* This is proved by rescaling. To prove the first claim, use Lemma 17 to find  $a, b \in \mathbb{Z}$  with  $a + b\sqrt{2} \in [x_0/\sqrt{2}, x_1/\sqrt{2}]$  and  $a - b\sqrt{2} \in [-y_1/\sqrt{2}, -y_0/\sqrt{2}]$ . Let  $a' = 2b$  and  $b' = a$ . Then we have  $a' + b'\sqrt{2} = \sqrt{2}(a + b\sqrt{2}) \in [x_0, x_1]$  and  $a' - b'\sqrt{2} = -\sqrt{2}(a - b\sqrt{2}) \in [y_0, y_1]$ , as desired. To prove the second claim, use the first claim to find  $a' + b'\sqrt{2} \in [x_0 - 1, x_1 - 1]$  and  $a' - b'\sqrt{2} \in [y_0 - 1, y_1 - 1]$ , with  $a'$  even; then let  $a'' = a' + 1$  and  $b'' = b'$ .  $\square$

## 6 Approximation up to $\epsilon$

As mentioned in Section 2, given  $\theta$  and  $\epsilon$ , we wish to find  $k \geq 0$  and  $u, t \in \mathbb{Z}[\omega]$  such that

$$U = \frac{1}{\sqrt{2}^k} \begin{pmatrix} u & -t^\dagger \\ t & u^\dagger \end{pmatrix}$$

satisfies  $\|U - R_z(\theta)\| \leq \epsilon$ . We now elaborate how to determine  $k, u,$  and  $t$ .

### 6.1 The $\epsilon$ -region

We first examine how to express the error  $\epsilon$  as a function of  $u$ ; this will make explicit the set of available  $u$  for a given  $\epsilon$ . Let  $z = x + yi = e^{-i\theta/2}$ . First note that, since both  $U$  and  $R_z(\theta)$  are  $2 \times 2$  unitary of determinant 1, the operator norm of  $U - R_z(\theta)$  coincides with  $1/\sqrt{2}$  of the Hilbert-Schmidt norm, which can therefore be calculated as follows. Let  $u' = \frac{1}{\sqrt{2}^k} u = a + bi$  and  $t' = \frac{1}{\sqrt{2}^k} t$ .

$$\|U - R_z(\theta)\|^2 = \frac{1}{2} \|U - R_z(\theta)\|_{\text{HS}}^2 = |u' - z|^2 + |t'|^2. \quad (22)$$

Using  $u'u'^\dagger + t't'^\dagger = 1$  and  $zz^\dagger = 1$ , we can further simplify this to:

$$|u' - z|^2 + |t'|^2 = (u' - z)(u' - z)^\dagger + t't'^\dagger = u'u'^\dagger - u'z^\dagger - zu'^\dagger + zz^\dagger + t't'^\dagger = 2 - 2\operatorname{Re}(u'z^\dagger) = 2 - 2\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \quad (23)$$

The error is therefore directly related to the dot product of  $u$  and  $z$ , regarded as vectors in  $\mathbb{R}^2$ . Writing  $\mathbf{z} = (x, y)^T$  and  $\mathbf{u} = (a, b)^T$ , we have

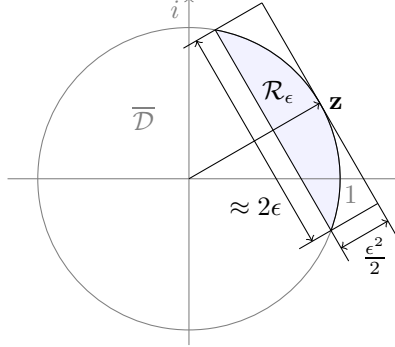
$$\|U - R_z(\theta)\|^2 \leq \epsilon^2 \iff 2 - 2\mathbf{u} \cdot \mathbf{z} \leq \epsilon^2 \iff \mathbf{u} \cdot \mathbf{z} \geq 1 - \frac{\epsilon^2}{2}. \quad (24)$$



Let us define the  $\epsilon$ -region for  $z$  to be the corresponding subset of the unit disk. Let  $\overline{\mathcal{D}} = \{\mathbf{u} \mid |\mathbf{u}|^2 \leq 1\}$  be the closed unit disk. Then

$$\mathcal{R}_\epsilon = \{\mathbf{u} \in \overline{\mathcal{D}} \mid \mathbf{u} \cdot \mathbf{z} \geq 1 - \frac{\epsilon^2}{2}\}. \quad (25)$$

The  $\epsilon$ -region is shown as a shaded region in the illustration below. Note that the width of this region is  $\frac{\epsilon^2}{2}$  at its widest point, and its length, to second order, is  $2\epsilon$ , and in any case greater than  $\sqrt{2}\epsilon$ .



In summary,  $\|U - R_z(\theta)\| \leq \epsilon$  if and only if  $\mathbf{u} \in \mathcal{R}_\epsilon$ .

From now on, it will be convenient to identify the complex plane with  $\mathbb{R}^2$ , i.e., we will regard  $\overline{\mathcal{D}}$  and  $\mathcal{R}_\epsilon$  as subsets of the complex numbers as well as  $\mathbb{R}^2$ .

## 6.2 Candidates

Assume, for the moment, that some suitable  $k \geq 1$  has been fixed. The problem of finding an  $\epsilon$ -approximation of  $R_z(\theta)$  can now be reduced to the following 2-step problem:

- (a) find  $u \in \mathbb{Z}[\omega]$  such that  $u' = u/\sqrt{2}^k$  is in the  $\epsilon$ -region, and
- (b) find  $t \in \mathbb{Z}[\omega]$  such that  $tt^\dagger + uu^\dagger = 2^k$ .

Recall from Section 4 that we have an efficient solution to (b) if  $\xi = 2^k - uu^\dagger$  satisfies the hypotheses of Theorem 12. We will quite literally leave one of these hypotheses, namely the primality of  $\mathcal{N}_{\sqrt{2}}(\xi)$ , to chance. However, we will arrange things such that the remaining hypotheses are satisfied by design. To that end, we will say that  $u'$  is a *candidate* if it satisfies all the required conditions, except possibly for the primality of  $\mathcal{N}_{\sqrt{2}}(\xi)$ . This is made precise in the following definition and lemma.

**Definition 20.** Let  $\epsilon > 0$ ,  $\theta \in \mathbb{R}$ , and  $k \geq 1$  be fixed. Let  $u' = u/\sqrt{2}^k$ , where  $u \in \mathbb{Z}[\omega]$ . Let  $\xi = x + y\sqrt{2} = 2^k - uu^\dagger$ . Then  $u'$  is called a *candidate* if  $u' \in \mathcal{R}_\epsilon$ ,  $x$  is odd,  $y$  is even,  $\xi \geq 0$ , and  $\xi^\bullet \geq 0$ . We say that  $u'$  is a *prime candidate* if, moreover,  $p = \xi\xi^\bullet$  is prime.

We will further restrict attention to candidates  $u' = u/\sqrt{2}^k$  where  $u$  is from the ring  $\mathbb{Z}[\sqrt{2}, i] \subseteq \mathbb{Z}[\omega]$ , i.e., of the form  $u = \alpha + \beta i$  with  $\alpha, \beta \in \mathbb{Z}[\sqrt{2}]$ .

**Lemma 21.** Let  $u' = u/\sqrt{2}^k$ , where  $u = \alpha + \beta i$  and  $\alpha, \beta \in \mathbb{Z}[\sqrt{2}]$ . Let us write  $\alpha = a + b\sqrt{2}$  and  $\beta = c + d\sqrt{2}$ . Then  $u'$  is a candidate if and only if the following three conditions hold:

- $u' \in \mathcal{R}_\epsilon$ ,
- $u'^\bullet \in \overline{\mathcal{D}}$ , and
- $a + c$  is odd.

*Proof.* With the given choice of notation, we have

$$\begin{aligned} \xi &= 2^k - uu^\dagger \\ &= 2^k - \alpha^2 - \beta^2 \\ &= (2^k - a^2 - 2b^2 - c^2 - 2d^2) - (2ab + 2cd)\sqrt{2}, \end{aligned} \quad (26)$$

hence  $x = 2^k - a^2 - 2b^2 - c^2 - 2d^2$  and  $y = 2(ab + cd)$ . Then  $x$  is odd if and only if  $a + c$  is odd. The condition that  $y$  is even is automatically satisfied. The condition  $\xi \geq 0$  is equivalent to  $uu^\dagger \leq 2^k$ , or equivalently  $u'u'^\dagger \leq 1$ , i.e.,  $u' \in \overline{\mathcal{D}}$ , which follows from  $u' \in \mathcal{R}_\epsilon$ . Similarly,  $\xi^\bullet \geq 0$  is equivalent to  $u'^\bullet \in \overline{\mathcal{D}}$ .  $\square$

### 6.3 Candidate selection

**Theorem 22.** *Let  $\epsilon > 0$  and  $\theta$  be fixed, and let  $k \geq C + 2\log_2(1/\epsilon)$ , where  $C = 3 + 2\log_2(1 + \sqrt{2})$ . Then there exists a set of at least  $n = \lfloor \frac{4\sqrt{2}}{\epsilon} \rfloor$  candidates  $u'$  satisfying the conditions of Lemma 21. Moreover, there is an efficient algorithm for computing a random candidate from this set.*

*Proof.* Define

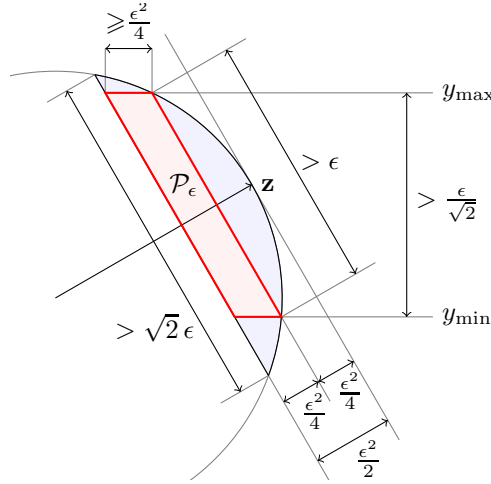
$$\delta = \sqrt{2}^k \frac{\epsilon^2}{8} \quad \text{and} \quad \Delta = \sqrt{2}^{k+1}. \quad (27)$$

We first note that  $\delta$  and  $\Delta$  satisfy the condition of Lemma 17. Indeed,

$$\delta\Delta = 2^k \frac{\sqrt{2}\epsilon^2}{8} \geq \frac{8(1+\sqrt{2})^2}{\epsilon^2} \frac{\epsilon^2}{8} = (1+\sqrt{2})^2. \quad (28)$$

In the following, we will assume, for convenience, that  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . This assumption is without loss of generality, because otherwise, we may simply rotate the  $\epsilon$ -region by multiples of  $90^\circ$  without changing the substance of the argument.

Now consider the line  $\mathbf{u} \cdot \mathbf{z} = 1 - \epsilon^2/4$ . It intersects the unit circle in two points at  $y$ -coordinates  $y_{\min}$  and  $y_{\max}$ , with  $y_{\max} - y_{\min} \geq \frac{\epsilon}{\sqrt{2}}$ , as shown in this figure:



Consider the parallelogram  $\mathcal{P}_\epsilon$  that is bounded by the lines  $\mathbf{u} \cdot \mathbf{z} = 1 - \epsilon^2/4$  and  $\mathbf{u} \cdot \mathbf{z} = 1 - \epsilon^2/2$ , and by the horizontal lines at  $y = y_{\min}$  and  $y = y_{\max}$ . As illustrated in the above figure,  $\mathcal{P}_\epsilon$  is entirely contained within the  $\epsilon$ -region. We will select candidates from within  $\mathcal{P}_\epsilon$ .

Let  $n = \lfloor \frac{4\sqrt{2}}{\epsilon} \rfloor$ , and define  $y_j = y_{\min} + \frac{j}{n}(y_{\max} - y_{\min})$  for  $j = 0, \dots, n$ , so that  $y_{\min} = y_0 < y_1 < \dots < y_n = y_{\max}$ . We note that

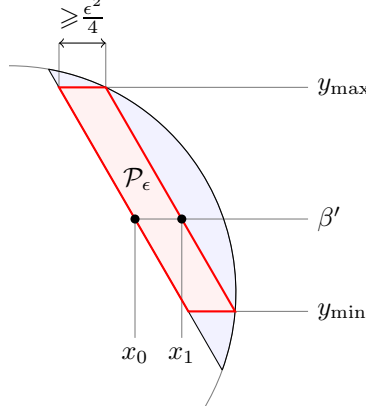
$$y_{j+1} - y_j = \frac{y_{\max} - y_{\min}}{n} > \frac{\epsilon}{\sqrt{2}} \frac{\epsilon}{4\sqrt{2}} = \frac{\epsilon^2}{8}. \quad (29)$$

Let  $I_j = [y_j, y_j + \frac{\epsilon^2}{8}]$  for  $j = 0, \dots, n-1$ . Then  $I_0, \dots, I_{n-1}$  are non-overlapping closed subintervals of  $[y_{\min}, y_{\max}]$  of size  $\frac{\epsilon^2}{8}$ . For each  $j = 0, \dots, n-1$ , we will find a candidate  $u' \in \mathcal{P}_\epsilon \cap (\mathbb{R} \times I_j)$  as follows.

First, use Lemma 17 to find  $\beta \in \mathbb{Z}[\sqrt{2}]$  such that  $\beta \in [\sqrt{2}^k y_j, \sqrt{2}^k (y_j + \frac{\epsilon^2}{8})]$  and  $\beta^\bullet \in [-\sqrt{2}^{k-1}, \sqrt{2}^{k-1}]$ . Note that these two intervals are of size  $\delta$  and  $\Delta$ , respectively, so the use of Lemma 17 is justified by (28). Let  $\beta' = \beta/\sqrt{2}^k \in I_j$ .

Because  $\beta' \in [y_{\min}, y_{\max}]$ , the line  $y = \beta'$  intersects the parallelogram  $\mathcal{P}_\epsilon$ , as shown in the figure below. Let

$x_0 = \min\{x \mid (x, \beta') \in \mathcal{P}_\epsilon\}$  and  $x_1 = \max\{x \mid (x, \beta') \in \mathcal{P}_\epsilon\}$ , and note that  $x_1 - x_0 \geq \frac{\epsilon^2}{4}$ .



Use Corollary 19 to find  $\alpha \in \mathbb{Z}[\sqrt{2}]$  such that  $\alpha \in [\sqrt{2}^k x_0, \sqrt{2}^k (x_0 + \frac{\epsilon^2}{4})]$  and  $\alpha^\bullet \in [-\sqrt{2}^{k-1}, \sqrt{2}^{k-1}]$ . Note that these two intervals are of size  $2\delta$  and  $\Delta$ , respectively, so that the use of Corollary 19 is again justified by (28). Furthermore, writing  $\alpha = a + b\sqrt{2}$  and  $\beta = c + d\sqrt{2}$ , Corollary 19 permits us to choose  $a$  odd if  $c$  is even, or vice versa.

Let  $\alpha' = \alpha/\sqrt{2}^k$ , and let  $u' = \alpha' + \beta'i$ . It is now trivial to show that  $u'$  is a candidate. Indeed, since  $\alpha' \in [x_0, x_1]$ , we have that  $u' \in \mathcal{P}_\epsilon \subseteq \mathcal{R}_\epsilon$  by construction. Also, note that, by construction,  $(\alpha'^\bullet, \beta'^\bullet) \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^2 \subseteq \overline{\mathcal{D}}$ , so that  $u'^\bullet \in \overline{\mathcal{D}}$ . Finally, we already remarked that  $a + c$  is odd.

Since we have found a distinct candidate for each  $j \in \{0, \dots, n-1\}$ , there exist at least  $n$  candidates; moreover, the construction of the  $j$ th candidate is clearly algorithmic.  $\square$

## 7 The main algorithm

### 7.1 Approximating a $z$ -rotation

We are now ready to put together the results of earlier sections to obtain an algorithm for approximating  $R_z(\theta)$  up to  $\epsilon$ , using only gates from the single-qubit Clifford+ $T$  group.

**Algorithm 23.** Inputs:  $0 < \epsilon \leq \frac{1}{2}$  and  $\theta \in \mathbb{R}$ . Let  $k = \lceil C + 2\log_2(1/\epsilon) \rceil$ , where  $C = 3 + 2\log_2(1 + \sqrt{2}) \approx 5.54$ , and let  $n = \lfloor \frac{4\sqrt{2}}{\epsilon} \rfloor$ . Use Theorem 22 to generate random candidates  $u' \in \frac{1}{\sqrt{2}^k} \mathbb{Z}[\omega]$ . For each candidate  $u' = u/\sqrt{2}^k$ , let  $\xi = 2^k - uu^\dagger \in \mathbb{Z}[\sqrt{2}]$  and attempt to use the method of Theorem 12 to find  $t \in \mathbb{Z}[\omega]$  with  $tt^\dagger = \xi$ . If this fails, continue with the next candidate. If it succeeds, the operator

$$U = \frac{1}{\sqrt{2}^k} \begin{pmatrix} u & -t^\dagger \\ t & u^\dagger \end{pmatrix}$$

is the desired  $\epsilon$ -approximation of  $R_z(\theta)$ . Use the Kliuchnikov-Maslov-Mosca exact synthesis algorithm [8] to convert  $U$  to a sequence of Clifford+ $T$  gates; optionally, reduce the sequence of gates to Matsumoto-Amano normal form [9]. Output: the sequence of gates.

**Remark 24.** It is not necessary to limit the generation of candidates to the particular set of  $n$  candidates identified in Theorem 22. Instead, one can simply generate candidates at random until a solution is found.

**Remark 25.** Although the method of Theorem 12 will usually only work when  $p = \xi\xi^\bullet$  is prime, Algorithm 23 requires no explicit primality test. Instead, one can simply perform the method of Theorem 12 under the optimistic assumption that  $p$  is prime. In the worst case, this may yield a  $t$  that is not a solution, but this can be easily checked after the fact. Some slight care is needed to ensure that the probabilistic algorithm from Remark 11, for finding a square root  $h$  of  $-1$  modulo  $p$ , does not get into an infinite loop when  $p$  is not prime. For this, it is sufficient to limit the number of iterations of this algorithm to some small number, say 1 or 2. If  $p$  is indeed prime, this will still yield  $h$  with high enough probability; in the worst case, some prime candidates will be unnecessarily rejected.

## 7.2 Approximating arbitrary gates

To approximate an arbitrary gate  $U \in SU(2)$ , first decompose it via Euler angles as  $U = R_z(\beta)R_x(\gamma)R_z(\delta) = R_z(\beta)H R_z(\gamma)H R_z(\delta)$ ; each rotation can then be approximated separately, say up to  $\epsilon/3$ .

**Remark 26.** If one includes, as we did, only global phases of the form  $\omega^j$  in the Clifford group, it is of course not possible to approximate unitary operators whose determinant is arbitrary, using only single-qubit Clifford+ $T$  gates. Since the determinant of every Clifford+ $T$  operator is a power of  $\omega$ , an operator  $U \in U(2)$  can be approximated to arbitrary  $\epsilon$  if and only if  $\det U$  is also a power of  $\omega$ . Otherwise,  $U$  can only be approximated up to a global phase. Had we included arbitrary global phases in the Clifford group, then of course all operators could be approximated.

## 8 Complexity analysis

### 8.1 Gate complexity

Since the matrix  $U$  constructed by Algorithm 23 has denominator exponent  $k$ , the approximation of a  $z$ -rotation uses  $T$ -count at most  $2k$ , or  $2C + 4\log_2(1/\epsilon)$ . The approximation of an arbitrary element of  $SU(2)$  requires  $T$ -count  $C' + 12\log_2(1/\epsilon)$ .

**Remark 27.** Rotations about other “easy” axes, such as  $x$ -rotations or  $y$ -rotations, can of course also be approximated with the same  $T$ -count as a  $z$ -rotation, as they differ from a  $z$ -rotation only by Clifford operators. More generally, rotations of the form  $VR_z(\theta)V^\dagger$ , where  $V$  is a fixed Clifford+ $T$ -operator, can be approximated with  $T$ -count  $K + 4\log_2(1/\epsilon)$ , where  $K$  is a constant depending only on  $V$ .

### 8.2 Time complexity

Most parts of the algorithm are computationally straightforward; for example, the generation of candidates, and the various ring operations required for Theorem 12, can all be done with integers of length  $O(k)$ . The arithmetic operations, such as addition, multiplication, division, etc., require no more than  $O(k^2)$  elementary steps each, and there are  $O(k)$  arithmetic operations to perform.

The dominant complexity question is how many candidates must be generated before a solution is found, and indeed, whether a solution will be found at all. Here, we must make an assumption about the distribution of primes:

**Hypothesis 28.** For a randomly chosen candidate, the probability that  $p = \xi\xi^\bullet$  is prime is asymptotically no smaller than for general odd numbers of comparable size.

We note that for a candidate  $u' = u/\sqrt{2}^k$ , we have

$$u'u'^\dagger \geq \left(1 - \frac{\epsilon^2}{2}\right)^2 \geq 1 - \epsilon^2, \quad (30)$$

and thus

$$\xi = 2^k - uu^\dagger = 2^k(1 - u'u'^\dagger) \leq 2^k\epsilon^2 \leq \frac{16(1 + \sqrt{2})^2}{\epsilon^2}\epsilon^2 = 16(1 + \sqrt{2})^2 \leq 100. \quad (31)$$

Also,  $\xi^\bullet = 2^k - u^\bullet u^{\bullet\dagger} \leq 2^k$ . It follows that  $p = \xi\xi^\bullet \leq 100 \cdot 2^k$ . By the prime number theorem, a randomly chosen odd  $p$  in this range has probability

$$P \approx \frac{2}{\ln(100 \cdot 2^k)} = \Omega(1/k) \quad (32)$$

of being prime; by Hypothesis 28, at least the same probability holds for a randomly chosen candidate. On the other hand, there are  $n = \lfloor \frac{4\sqrt{2}}{\epsilon} \rfloor = O(\sqrt{2}^k)$  candidates available, so asymptotically, a prime will be found with certainty. The expected number of top-level iterations of Algorithm 23 until a solution is found is  $O(k)$ . Therefore, under Hypothesis 28, the overall runtime of the algorithm is no more than  $O(k^4) = O(\log^4(1/\epsilon))$ . A sharper bound could probably be derived by a more sophisticated analysis.

We also note that, due to its randomized nature, and because each candidate is chosen independently, the algorithm can easily be parallelized.

### 8.3 Seeding

While the above discussion concerns the asymptotic case, one may wonder whether for *particular* values of  $\epsilon$  and  $\theta$  it may happen, by unlucky coincidence, that none of the available candidates are prime. In practice, this never seems to be the case, as primes are always found quite quickly. However, in theory, we may avoid this situation by the method of *seeding*: Instead of approximating  $R_z(\theta)$  directly, we may choose a random angle (“seed”)  $\phi$ , approximate both  $R_z(\phi)$  and  $R_z(\theta - \phi)$  to within  $\epsilon/2$ , and finally compute  $R_z(\theta)$  as  $R_z(\phi)R_z(\theta - \phi)$ . If a particular seed seems to yield no prime candidates in a given number of iterations, one can simply try a different seed.

The seeding method for  $z$ -rotations has the disadvantage that it approximately doubles the  $T$ -count, from  $2C + 4\log_2(1/\epsilon)$  to  $4C + 8\log_2(1/\epsilon)$ .

In the approximation of an arbitrary gate  $U \in SU(2)$ , a different seeding method can be used, which only increases the asymptotic  $T$ -count by a small additive constant. Namely, let the seed be some randomly chosen Clifford+ $T$  operator  $V$  of fixed and relatively small  $T$ -count. We can then use the algorithm to approximate  $UV^{-1}$ , and multiply the final result by  $V$ . The gate complexity, in this case, is unchanged at  $K + 12\log_2(1/\epsilon)$ .

It must be emphasized, once again, that seeding is only of theoretical interest, to ensure expected termination of the algorithm under a hypothetical worst-case scenario. In practice, it does not seem to be required.

## 9 Lower bounds

As mentioned in the introduction,  $K + 3\log_2(1/\epsilon)$  is an easy lower bound for the  $T$ -count of a Clifford+ $T$  approximation of some arbitrary operator in  $SU(2)$  up to  $\epsilon$ , for some constant  $K$ . Specifically, Matsumoto and Amano showed that there are precisely  $192 \cdot (3 \cdot 2^n - 2)$  distinct single-qubit Clifford+ $T$ -circuits of  $T$ -count at most  $n$  [9]. Since  $SU(2)$  is a 3-dimensional manifold, it requires  $\Omega(1/\epsilon^3)$  epsilon-balls to cover. The resulting inequality

$$192 \cdot (3 \cdot 2^n - 2) \geq \frac{c}{\epsilon^3}, \quad (33)$$

immediately implies

$$n \geq K + 3\log_2(1/\epsilon). \quad (34)$$

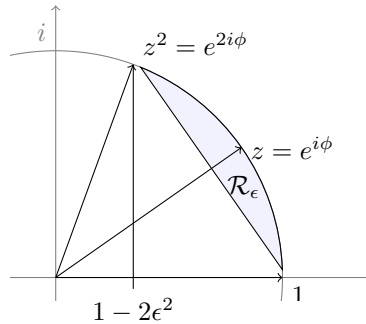
This means that there is some universal constant  $K$  such that for every  $\epsilon$ , there exists some operator  $U \in SU(2)$  that cannot be approximated up to  $\epsilon$  with  $T$ -count less than  $K + 3\log_2(1/\epsilon)$ . In fact, (33) implies that the proportion (in the Haar measure, say) of operators that can be approximated with  $T$ -count  $K + 3\log_2(1/\epsilon) - k$  is at most  $O(\frac{1}{2^k})$ , so that “most” operators require  $T$ -counts that are close to or above the lower bound.

However, like all lower bounds, this must be understood with a grain of salt. For example, the set of  $z$ -rotations only forms a 1-dimensional submanifold of  $SU(2)$ , so it is not a priori clear that the lower bound of  $K + 3\log_2(1/\epsilon)$  also applies to  $z$ -rotations.

We now prove a sharper lower bound, which applies to  $z$ -rotations in particular.

**Theorem 29.** *Let  $K' = -9$ . Then for all  $\epsilon > 0$ , there exists an angle  $\theta$  such that every  $\epsilon$ -approximation of  $R_z(\theta)$  by a product of Clifford+ $T$  operators requires  $T$ -count at least  $K' + 4\log_2(1/\epsilon)$ .*

*Proof.* Let  $\epsilon$  be given. If  $\epsilon \geq 1/2$ , then  $K' + 4\log_2(1/\epsilon)$  is negative, so there is nothing to show. Assume, therefore, that  $\epsilon < 1/2$ . Let  $\phi = \sin^{-1} \epsilon$ , with  $0 < \phi < \pi/6$ . Let  $\theta = -2\phi$ , so that  $z = e^{-i\theta/2} = e^{i\phi}$ . We note the shape of the  $\epsilon$ -region for  $z$ :



Identifying complex number with vectors in  $\mathbb{R}^2$  as before, we estimate the dot product

$$\mathbf{1} \cdot \mathbf{z} = \cos \phi = \sqrt{1 - \epsilon^2} < \sqrt{1 - \epsilon^2 + \frac{\epsilon^4}{4}} = 1 - \frac{\epsilon^2}{2}. \quad (35)$$

It follows that neither 1 nor  $z^2$  is in the  $\epsilon$ -region. We further note that the  $x$ -coordinate of  $z^2$  is  $\cos 2\phi = 1 - 2\sin^2 \phi = 1 - 2\epsilon^2$ . So for all  $x + iy \in \mathcal{R}_\epsilon$ , we have

$$1 - 2\epsilon^2 < x < 1. \quad (36)$$

Now consider some  $\epsilon$ -approximation

$$U = \frac{1}{\sqrt{2}^k} \begin{pmatrix} u & s \\ t & v \end{pmatrix} \quad (37)$$

of  $R_z(\theta)$  with denominator exponent  $k$ , i.e., where  $u, t, s, v \in \mathbb{Z}[\omega]$ . By the same reasoning as in Section 6.1, it follows that  $u' = u/\sqrt{2}^k$  is in the  $\epsilon$ -region. Moreover since  $U^\bullet$  is also unitary, we have  $u'^\bullet \in \overline{\mathcal{D}}$ . We can write

$$u' = \frac{1}{\sqrt{2}^k} (a\omega^3 + b\omega^2 + c\omega + d) = \frac{1}{\sqrt{2}^{k+1}} (d\sqrt{2} + c - a) + (b\sqrt{2} + c + a)i = \frac{1}{\sqrt{2}^{k+1}} (\alpha + \beta i),$$

where  $\alpha, \beta \in \mathbb{Z}[\sqrt{2}]$ . Since  $u' \in \mathcal{R}_\epsilon$ , from (36), we have

$$\sqrt{2}^{k+1} (1 - 2\epsilon^2) < \alpha < \sqrt{2}^{k+1}. \quad (38)$$

Also, from  $u'^\bullet \in \overline{\mathcal{D}}$ , we have

$$\alpha^\bullet \in [-\sqrt{2}^{k+1}, \sqrt{2}^{k+1}]. \quad (39)$$

It follows that the constraints

$$(\gamma, \gamma^\bullet) \in [\sqrt{2}^{k+1} (1 - 2\epsilon^2), \sqrt{2}^{k+1}] \times [-\sqrt{2}^{k+1}, \sqrt{2}^{k+1}] \quad (40)$$

have at least two different solutions in  $\mathbb{Z}[\sqrt{2}]$ , namely  $\gamma = \alpha$  and  $\gamma = \sqrt{2}^{k+1}$ . Setting  $\delta = \sqrt{2}^{k+1} 2\epsilon^2$  and  $\Delta = 2\sqrt{2}^{k+1}$ , we have by Lemma 16:

$$1 \leq \delta \Delta = 2^{k+3} \epsilon^2, \quad (41)$$

or equivalently,

$$k \geq \log_2\left(\frac{1}{8\epsilon^2}\right) = -3 + 2\log_2\left(\frac{1}{\epsilon}\right). \quad (42)$$

On the other hand, it is a fact that every single-qubit Clifford+ $T$  operator  $U$  of  $T$ -count  $n$  can be written with denominator exponent  $k$ , where

$$k \leq \frac{n+3}{2}. \quad (43)$$

This can be shown by an easy induction on the Matsumoto-Amano normal form of  $U$ . Putting together (42) and (43), we get

$$n \geq -9 + 4\log_2(1/\epsilon), \quad (44)$$

which is the desired result.  $\square$

**Remark 30.** Unlike the lower bound (34), which applies to *typical* operators, the lower bound (44) only applies to carefully chosen worst-case operators. It is plausible that for any fixed  $\epsilon$ , approximations of order (34) can be achieved for most angles  $\theta$ . Moreover, it is also plausible that for any fixed  $\theta$ , approximations of order (34) can be achieved as  $\epsilon \rightarrow 0$ . A variation of Algorithm 23 that could potentially achieve this is sketched in Section 9.1, but the details are left for future work.

## 9.1 Overclocking

It is in the nature of Algorithm 23 that  $k$  is chosen at the very beginning; the final sequence of gates will have  $T$ -count very close to  $2k$ . This behavior is different from that of search-based algorithms, which would typically try shorter decompositions first, and then gradually move to longer ones.

In practice, many of the approximations and estimates in the above proofs are conservative; it is often possible to choose a smaller  $k$  than that prescribed by the algorithm, and still obtain a decomposition. I refer to this technique as “overclocking” the algorithm, by analogy with the practice of running microprocessors at higher clock speeds than they were designed for, and hoping for the best.

Let us first consider the effect of overclocking by a small additive constant, i.e., decreasing the value of  $C$  in Algorithm 23 by some fixed amount, independently of  $\epsilon$ . This decreases the width of the  $\epsilon$ -region by a fixed multiplicative factor, which means that some choices of  $\beta$  (in Theorem 22) will no longer yield a successful solution



## 11 Conclusions

We have given the most efficient algorithm to date for decomposing an element of  $SU(2)$  into a product of Clifford+ $T$  gates, up to arbitrarily small  $\epsilon$ . The algorithm performs well both in theory and in practice, easily achieving decompositions up to  $\epsilon = 10^{-100}$  using  $T$ -counts of less than 1400.

Like the recent ancilla-based algorithm by Kliuchnikov, Maslov, and Mosca [7], our algorithm is based on solving a Diophantine equation. Although both algorithms achieve the same optimal asymptotic big-O complexity, the gate decompositions resulting from our algorithm are shorter (by a constant, but non-negligible factor) than those that can be achieved by the Kliuchnikov-Maslov-Mosca approximation algorithm.

## 12 Acknowledgements

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